Chapter 3 Additional Questions

21. i) Let u and v be inverse arithmetic functions, so $u * v = \delta$. Let F and G be functions $F, G : \mathbb{R} \to \mathbb{C}$. Prove, using Möbius Inversion, that

$$G(x) = \sum_{m \le x} u(m) F\left(\frac{x}{m}\right) \tag{20}$$

if, and only if,

$$\sum_{n \le x} v(n) G\left(\frac{x}{n}\right) = F(x). \tag{21}$$

Hint: Assume (21) and insert this expression for G into the sum in (22), and vice verse.

22. i) Prove that for von Mangoldt's function Λ we have

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d, \tag{22}$$

for $n \ge 2$. This can be written as $\Lambda = -1 * \mu\ell$, where $\mu\ell(n) = \mu(n) \log n$.

ii) Deduce that if $\gcd(m, n) = 1$ then

$$\Lambda(mn) = \delta(n) \Lambda(m) + \delta(m) \Lambda(n),$$

where $\delta(n) = 1$ if $n = 1$, 0 otherwise.

Hint for ii. Recall that if $\gcd(m, n) = 1$ then the divisors d of mn are in one-to-one relation with pairs of divisors $d_1|m$ and $d_2|n$. Further we also have $\gcd(d_1, d_2) = 1$ in which case $\mu(d_1 d_2) = \mu(d_1) \mu(d_2)$. Use this recollection in the convolution $\Lambda = \mu * \ell$ evaluated at mn.

23. Assume F is an arithmetic function and $G(n) = \sum_{d|n} F(d)$, in which case $F(n) = \sum_{d|n} \mu(d) G(n/d)$ by Möbius Inversion.

i) Prove that

$$\sum_{m|n} \mu(m) \log\left(\frac{n}{m}\right) G\left(\frac{n}{m}\right) = F(n) \log n + \sum_{d|n} \Lambda(d) F\left(\frac{n}{d}\right).$$

Hint start with $\log(n/m) = \log n - \log m$ and use Question 22.

ii) Generalise $\Lambda = \mu * \ell$ by defining $\Lambda_k = \mu * \ell^k$ for all $k \geq 0$. Prove that

$$\Lambda_{k+1} = \Lambda_k \ell + \Lambda * \Lambda_k, \tag{23}$$

i.e.

$$\Lambda_{k+1}(n) = \Lambda_k(n) \log n + \Lambda * \Lambda_k(n)$$

for all $n \geq 1$.

The reason for this question is that one can show that Λ_k is non-zero only on integers with at most k distinct prime divisors, i.e. $\omega(n) \leq k$. Thus using Λ_k it may be possible to prove results concerning such integers.

24. A special case of Question 21 with $u = 1$ and $v = \mu$, states that if F is a function on $[1, \infty)$ and $G(x) = \sum_{n \leq x} F(x/n)$, then

$$F(x) = \sum_{n \leq x} \mu(n) G\left(\frac{x}{n}\right). \tag{24}$$

Prove the ***Tatuzawa-Iseki*** Identity

$$\sum_{m \leq x} \mu(m) \log\left(\frac{x}{m}\right) G\left(\frac{x}{m}\right) = F(x) \log x + \sum_{n \leq x} \Lambda(n) F\left(\frac{x}{n}\right).$$

This identity is seen in some elementary proofs of the Prime Number Theorem (i.e. proofs that do not use complex analysis).

Hint On the left hand side write $\log(x/m) = \log x - \log m$. On one of the terms use (25), on the other substitute in for $G(x/m)$ and rearrange the sums.

25. The function $Q_2(n)$ could be written as $|\mu|(n)$ defined as $|\mu(n)|$. Thus we could look at the general $|\mu_k|$ for $k \geq 1$. Here $|\mu_k|$ is multiplicative and satisfies

$$|\mu_k|(p^a) = \begin{cases} 1 & \text{if } a = 0 \text{ or } k \\ 0 & \text{otherwise.} \end{cases}$$

i) Prove that

$$D_{|\mu_k|}(s) = \frac{\zeta(ks)}{\zeta(2ks)}$$

for $\operatorname{Re} s > 1/k$.

ii) Why does this suggest $\mu_k * |\mu_k| = \mu_{2k}$?

Prove this by showing equality on prime powers

iii) Deduce that $Q_k * |\mu_k| = Q_{2k}$.

iv) What result from the problems for chapter 4 suggests $q_2 = sq * |\mu_3|$?

Prove this by showing equality on prime powers.

26. i) Prove that for any two arithmetic functions f and g we have

$$\lambda\,(f * g) = (\lambda f) * (\lambda g)$$

where λ is Liouville's function and

$$2^{\Omega}\,(f * g) = \left(2^{\Omega} f\right) * \left(2^{\Omega} g\right).$$

Here we are looking at products of functions as well as convolutions.

ii) Deduce that

$$\lambda d = \lambda * \lambda \quad \text{and} \quad \lambda d^2 = \lambda * \lambda * \lambda * \mu,$$

and thus

$$D_{\lambda d}(s) = \left(\frac{\zeta(2s)}{\zeta(s)}\right)^2 \quad \text{and} \quad D_{\lambda d^2}(s) = \frac{\zeta^3(2s)}{\zeta^4(s)}$$

for $\operatorname{Re} s > 1$.

27. From Question 5

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \zeta(2s).$$

From Question 9

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} \sum_{n=1}^{\infty} \frac{\lambda(n)\, 2^{\omega(n)}}{n^s} = 1.$$

From Question 11

$$\sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} \sum_{n=1}^{\infty} \frac{\lambda(n)\, d(n^2)}{n^s} = \zeta(2s).$$

From Question 13

$$\sum_{n=1}^{\infty} \frac{d^2(n)}{n^s} \sum_{n=1}^{\infty} \frac{\lambda(n)\, d^2(n)}{n^s} = \zeta^2(2s)\,.$$

These results suggest $1 * \lambda = sq$ (seen already in Question 7iii) along with

i) $2^{\omega} * \lambda 2^{\omega} = \delta$, hence $\lambda 2^{\omega}$ is the inverse of 2^{ω},

ii) $g * \lambda g = sq$ where $g\,(n) = d\,(n^2)$,

iii) $d^2 * \lambda d^2 = sq * sq$.

Prove Part i by checking equality on prime powers.

Deduce ii and iii using known results on convolutions of functions.

28. Questions 5, 9, 13 and 15 all started by taking a multiplicative arithmetic function, forming its Dirichlet series and then factorising the associated Euler product into products and quotients of the Riemann zeta function.

An alternative method is to start with the product and quotient of Riemann zeta functions, and multiply out all their Euler products to find one Euler Product of the form

$$\prod_p \left(1 + \frac{c_1}{p^s} + \frac{c_2}{p^{2s}} + \frac{c_3}{p^{3s}} + ... + \frac{c_r}{p^{rs}} + ...\right).$$

Here the numbers c_i will not depend on p. If we multiply this out we get a Dirichlet Series $D_f\,(s)$ and the associated function is multiplicative and given by

$$f(n) = \prod_{p^a || n} c_a.$$

Do this in the following cases, multiply out the Euler products for the Riemann zeta functions and find the associated arithmetic function.

i) $\dfrac{\zeta(3s)}{\zeta(s)}$, ii) $\dfrac{\zeta(2s)}{\zeta(3s)}$, iii) $\dfrac{\zeta^3(s)}{\zeta(3s)}$.

29. The result $Q_k = 1 * \mu_k$ of Example 3.31 could be compared with an earlier result, Question 10, $2^\omega = d * \mu_2$. This suggests looking at $d * \mu_k$ for $k \geq 2$.

Describe the function $d * \mu_k$.

Hint Because this is a multiplicative function it suffices to describe the value of this function on prime powers.

A sort of truncated divisor function.

30. Define $\phi_\nu = \mu * j^\nu$ for $\nu \in \mathbb{C}$. Show that

$$\phi_\nu(n) = n^\nu \prod_{p|n} \left(1 - \frac{1}{p^\nu}\right).$$

31. We have seen in Question 16 that, because $\sigma = 1 * j$, then

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\,\zeta(s-1)$$

for $\operatorname{Re} s > 1$. Give an *alternative* proof by looking at the Euler Product of the left hand side.

Hint Write

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \prod_p \left(\sum_{a=0}^{\infty} \frac{\sigma(p^a)}{p^{as}}\right),$$

and

$$\sigma(p^a) = \sum_{m=0}^{a} p^m.$$

The problem then becomes one of interchanging summations.

32. In Question 29 we have seen a sort of truncated divisor function given, on prime powers as

$$f_k(p^a) = \begin{cases} a+1 & \text{if } a \leq k-1, \\ k & \text{if } a \geq k, \end{cases}$$

and defined to be multiplicative on all integers.

By looking at its Euler Product, write

$$\sum_{n=1}^{\infty} \frac{f_k(n)}{n^s}$$

in terms of the Riemann zeta function.

This is Question 16 in reverse, there you started with the expression in terms of the Riemann zeta function and found the function f_k.

Hint When you write this as a Euler Product and write $y = 1/p^s$ you will need to sum a series of the form

$$S = 1 + 2y + 3y^2 + 4y^3 + ... + ky^{k-1} + ky^k + ky^{k+1} + \cdots$$

Consider $S - yS$.

33. *Results from this question are used in an Addition Question on the next Problem Sheet.*

i) Prove, by looking at the Euler Product of the Dirichlet Series, that

$$\sum_{n=1}^{\infty} \frac{Q_2(n)}{\phi(n)\, n^s}$$

converges for $\operatorname{Re} s > 0$.

ii) Show that

$$\sum_{n=1}^{\infty} \frac{Q_2(n)}{\phi(n)\, n} = \frac{\zeta(2)\, \zeta(3)}{\zeta(6)}.$$

(The constant is approximately 1.943596436... .)